



Existence results for nonlinear problems with ϕ -Laplacian

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Abstract. Using the barrier strip argument, we obtain the existence of solutions for the nonlinear boundary value problem

$$(\phi(u'))' = f(t, u, u'), \quad u(0) = A, \quad u'(1) = B,$$

where ϕ is an increasing homeomorphism.

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1 Introduction

The purpose of this article is to obtain some existence results for nonlinear problems of the form

$$(\phi(u'))' = f(t, u, u'), \quad t \in [0, 1], \quad (1.1)$$

$$u(0) = A, \quad u'(1) = B, \quad (1.2)$$

where $f: [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous and $\phi: (-\infty, \infty) \rightarrow \mathbb{R}$ is an increasing homeomorphism. Problems of this form are called classical.

A typical classical model is the well-known p -Laplacian equation

$$(\phi_p(u'))' = f(t, u, u'), \quad t \in [0, 1]$$

where $\phi_p(s) := |s|^{p-2}s$ ($p > 1$). Various two-point boundary value problems containing this operator have received a lot of attention lately with respect to existence of solutions, see for example, [10, 13, 14, 16, 20] and the references therein. The key condition in those works relies on a growth restriction on f .

In 1994, Kelevedjiev [18], using the topological transversality theorem and the barrier strip argument, studied the nonlinear non-homogeneous problem

$$u'' = f(t, u, u'), \quad t \in [0, 1] \quad (1.3)$$

$$u(0) = A, \quad u'(1) = B, \quad (1.4)$$

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and obtained the following theorem.

Theorem A. *Let $f: [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous, and let there exist constants L_i , $i = 1, 2, 3, 4$, such that $L_2 > L_1 \geq B$, $L_3 < L_4 \leq B$,*

$$f(t, u, v) \geq 0, \quad (t, u, p) \in [0, 1] \times \mathbb{R} \times [L_1, L_2], \quad (1.5)$$

and

$$f(t, u, v) \leq 0, \quad (t, u, p) \in [0, 1] \times \mathbb{R} \times [L_3, L_4]. \quad (1.6)$$

Then problem (1.3)–(1.4) has at least one solution.

(1.5) and (1.6) are called *barrier strip conditions*. They control the behavior of u' on $[0, 1]$. Depending on the sign of $f(t, u, u')$ the curve of $u'(t)$ on $[0, 1]$ crosses the strips $[0, 1] \times [L_1, L_2]$ and $[0, 1] \times [L_3, L_4]$ not more than once. Therefore, the assumptions (1.5) and (1.6) are sufficient conditions to obtain an a priori bound for $u(t)$ and $u'(t)$.

Very recently, Kelevedjiev and Tersian [19] generalized Theorem A to the following boundary value problem with p -Laplacian

$$(\phi_p(u'))' = f(t, u, u'), \quad u(0) = A, \quad u'(1) = B;$$

the existence of C^2 -solution is proved under barrier strip conditions similar to (1.5) and (1.6). This work raises the following question: “Can we replace the p -Laplacian operator by the more general increasing homeomorphism $(\phi(u'))'$?”. In the present paper, answering these questions in the affirmative, we extend Theorem A to the case of ϕ -Laplacian. More precisely, we prove the following theorem.

Theorem 1.1. *Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be classical and an increasing homeomorphism. Assume that (1.5) and (1.6) are fulfilled. Then (1.1)–(1.2) has at least one solution in $C^1[0, 1]$.*

The rest of the paper is organized as follows. In Section 2, we state some notations and the topological transversality theorem, which will be crucial in the proof of our main result. Section 3 is devoted to the preparation of *a priori* bounds for the possible solutions of a suitable family of problems, and finally we will prove the main result.

For other results concerning the ϕ -Laplacian operator we refer the reader to [1–9, 11, 12, 17].

2 Notations and fixed point theorem

As usual, $C[0, 1]$ is the Banach space of continuous functions defined on $[0, 1]$ endowed with the norm $\|\cdot\|_0$, and $C^1[0, 1]$ is Banach space of continuously differentiable functions defined on $[0, 1]$ endowed with the norm $\|u\|_1 = \max\{\|u\|_0, \|u'\|_0\}$.

Let Y be a convex subset of Banach space E and $U \subset Y$ be open in Y . Let $L_{\partial U}(\bar{U}, Y)$ be a set of compact maps from \bar{U} to Y which are fixed points free on ∂U ; here, as usual, \bar{U} and ∂U are the closure of U and the boundary of U in Y , respectively.

A map F in $L_{\partial U}(\bar{U}, Y)$ is *essential* if every map G in $L_{\partial U}(\bar{U}, Y)$ such that $G|_{\partial U} = F|_{\partial U}$ has a fixed point in U . It is clear, in particular, every essential map has a fixed point in U .

Theorem B (Topological transversality theorem, [15]). *Let Y be a convex subset of a Banach space E and $U \subset Y$ be open. Assume that*

(a) $F, G: U \rightarrow Y$ are compact maps;

(b) $G \in L_{\partial U}(\overline{U}, Y)$ is essential;

(c) $H(u, \lambda)$, $\lambda \in [0, 1]$, is a compact homotopy joining F and G ; i.e.,

$$H(u, 1) = F(u), \quad H(u, 0) = G(u);$$

(d) $H(u, \lambda)$, $\lambda \in [0, 1]$, is fixed point free on ∂U .

Then $H(u, \lambda)$, $\lambda \in [0, 1]$, has at least one fixed point in U and in particular there is a $u_0 \in U$ such that $u_0 = F(u_0)$.

Theorem C. Let $l \in U$ be fixed and $F \in L_{\partial U}(\overline{U}, Y)$ be the constant map $F(u) = l$ for $u \in U$, then F is essential.

3 Auxiliary results, proofs of the main results

For $\lambda \in [0, 1]$, we consider the family of boundary value problems

$$(\phi(u'))' = \lambda f(t, u, u'), \quad t \in [0, 1], \quad (3.1)$$

$$u(0) = A, \quad u'(1) = B, \quad (3.2)$$

Our first auxiliary result gives an a priori bound for the solutions of the problem (3.1)–(3.2).

Lemma 3.1. Let ϕ be an increasing homeomorphism, (1.5) and (1.6) hold and $u \in C^1[0, 1]$ be a solution of the problem (3.1)–(3.2). Then there exists a constant M independent of λ and u such that

$$\|u\|_1 < M.$$

Proof. Suppose the set

$$S_0 = \{t \in [0, 1] : L_1 < u'(t) \leq L_2\} \quad \text{and} \quad S_1 = \{t \in [0, 1] : L_3 \leq u'(t) < L_4\}$$

are not empty. Let $t_0 \in S_0$, $t_1 \in S_1$ be fixed. Assume that there are $t'_0 \in (t_0, 1]$ and $t'_1 \in (t_1, 1]$ such that

$$u'(t'_0) < u'(t_0), \quad u'(t'_1) > u'(t_1). \quad (3.3)$$

The continuity of $u'(t)$ allows us to take t'_0 and t'_1 correspondingly from $(t_0, 1] \cap S_0$ and $(t_1, 1] \cap S_1$. On the other hand, from (1.5) and (1.6) we have respectively

$$(\phi(u'))' = \lambda f(t, u, u') \geq 0 \quad \text{for } t \in S_0 \quad \text{and} \quad (\phi(u'))' = \lambda f(t, u, u') \leq 0 \quad \text{for } t \in S_1$$

and, since ϕ is increasing, we obtain that $u'(t)$ is monotone increasing for $t \in S_0$ and monotone decreasing for $t \in S_1$. Thus,

$$u'(t'_0) \geq u'(t_0), \quad u'(t'_1) \leq u'(t_1).$$

This contradicts (3.3). Consequently

$$u'(t) \geq u'(t_0) \quad \text{for } t \in (t_0, 1], \quad u'(t) \leq u'(t_1) \quad \text{for } t \in (t_1, 1],$$

and in particular

$$u'(1) \geq u'(t_0) > L_1 \geq B, \quad u'(1) \leq u'(t_1) < L_4 \leq B.$$

The contradiction obtained shows that S_0 and S_1 are empty. Since $u' \in C[0,1]$, then $L_4 \leq u'(t) \leq L_1$ for $t \in [0,1]$, i.e.,

$$|u'(t)| \leq \max\{|L_1|, |L_4|\}, \quad t \in [0,1].$$

On the other hand for each $t \in [0,1]$, we have

$$u(t) - u(0) = \int_0^t u'(s) ds,$$

which provides

$$|u(t)| \leq M_2 \quad \text{for } t \in [0,1],$$

where $M_2 = M_1 + |A|$, $M_1 = \max\{|L_1|, |L_4|\}$.

So, we obtained that each solution u of (3.1)–(3.2), satisfies

$$\|u\|_1 < M := \max\{M_1, M_2\} + 1,$$

where M is independent of λ and u . This completes the proof of Lemma 3.1. \square

Now, we introduce the set $C_{BC}^1[0,1] = \{u \in C^1[0,1] : u(0) = A, u'(1) = B\}$ and the operator $L: C_{BC}^1[0,1] \rightarrow C[0,1]$ defined by

$$Lu = (\phi(u'))'.$$

Introduce also the operator $K: C[0,1] \rightarrow C_{BC}^1[0,1]$ defined by

$$Kv = A + \int_0^t \phi^{-1} \left[\int_1^\tau v(s) ds + \phi(B) \right] d\tau.$$

Lemma 3.2. *The operator $K: C[0,1] \rightarrow C_{BC}^1[0,1]$ is well-defined and continuous.*

Proof. It is clear that for each $v \in C[0,1]$, the function $h(t) := \int_1^t v(s) ds + \phi(B)$ is continuous for $t \in [0,1]$. Since ϕ is an increasing homeomorphism, so $\phi^{-1}(h(t))$ is also continuous for $t \in [0,1]$.

Thus $(Kv)'(t) = \phi^{-1}(h(t))$ is in $C[0,1]$. Finally, it is easy to check that

$$(Kv)(0) = A, \quad (Kv)'(1) = B,$$

which means $(Kv)(t) \in C_{BC}^1[0,1]$. The continuity of K follows from the continuity of $A + \int_0^t \phi^{-1}[\int_1^\tau v(s) ds + \phi(B)] d\tau$ on $[0,1]$. \square

Lemma 3.3. *The operator K is the inverse operator of L .*

Proof. Clearly, each function $u \in C_{BC}^1[0,1]$ has a unique $v = Lu \in C[0,1]$. Also, each function $v \in C[0,1]$ has a unique inverse image $u \in C_{BC}^1[0,1]$ of the form

$$u = A + \int_0^t \phi^{-1} \left[\int_1^\tau v(s) ds + \phi(B) \right] d\tau,$$

which is the solution of the boundary value problem

$$\begin{aligned} (\phi(u'))' &= v(t), & t \in [0, 1], \\ u(0) &= A, & u'(1) = B. \end{aligned}$$

So, the operator L is one-to-one. Further, to show that K is an invertible map, let $Lu = v$, i.e., $\phi(u')' = v$. Then

$$\begin{aligned} Kv = K(Lu) &= A + \int_0^t \phi^{-1} \left[\int_1^\tau (\phi(u'(s)))' ds + \phi(B) \right] d\tau \\ &= A + \int_0^t \phi^{-1} [\phi(u'(\tau)) - \phi(u'(1)) + \phi(B)] d\tau \\ &= A + \int_0^t u'(\tau) d\tau \\ &= u(t). \end{aligned}$$

□

Proof of Theorem 1.1. At first, we introduce the set

$$U = \left\{ u \in C_{BC}^1[0, 1] : \|u\|_1 < M \right\}.$$

According to Lemma 3.1, all solutions of problem (3.1)–(3.2) are interior points of U . Introduce also the map

$$N: C^1[0, 1] \rightarrow C[0, 1], \quad \text{defined by} \quad (Nu)(t) = f(t, u(t), u'(t)),$$

for $t \in [0, 1]$ and $u(t) \in U$.

Now, we consider the homotopy

$$H_\lambda: \bar{U} \times [0, 1] \rightarrow C_{BC}^1[0, 1],$$

defined by $H(u, \lambda) \equiv H_\lambda(u) = \lambda KN(u) + (1 - \lambda)l$, where $l = Bt + A$ is the unique solution of

$$\begin{aligned} (\phi(u'))' &= 0, & t \in [0, 1], \\ u(0) &= A, & u'(1) = B. \end{aligned}$$

Since K and N are continuous, using the Arzelà–Ascoli theorem it is not difficult to see that K is compact. Therefore the homotopy $H_\lambda: \bar{U} \times [0, 1] \rightarrow C_{BC}^1[0, 1]$ is compact. According to Lemma 3.1, $H(u, \lambda)$, $\lambda \in [0, 1]$, is fixed point free on ∂U . Besides, $H_0(u) \equiv l$, $\forall u \in \bar{U}$; i.e., it is a constant map and so is essential, by Theorem C.

So, by Theorem B we get the map $H_1(u)$ has a fixed point in U . It is easy to see that it is a solution of the boundary value problem (3.1)–(3.2) obtained for $\lambda = 1$ and, what is the same, of (1.1)–(1.2). □

Example 3.4. Consider the boundary value problem

$$\begin{aligned} (\phi(u'))' &= u'^2 - 4u' + 3, & t \in [0, 1], \\ u(0) &= 0, & u'(1) = B. \end{aligned}$$

where $2 < B < 3$ and $\phi(s) = \frac{s^3}{1+s^2}$.

It is not difficult to verify that ϕ is an increasing homeomorphism, $f(t, u, u') = u'^2 - 4u' + 3$, has two simple zeros 1 and 3.

So, we can choose $L_1 = 4$, $L_2 = 5$, $L_3 = \frac{5}{4}$, $L_4 = \frac{3}{2}$ to see (1.5)–(1.6) hold and so the considered problem has at least one solution.

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